

# Mutations of the cluster algebra of type $A_1^{(1)}$ and the periodic discrete Toda lattice

Atsushi Nobe

Department of Mathematics, Faculty of Education, Chiba University,  
1-33 Yayoi-cho Inage-ku, Chiba 263-8522, Japan  
nobe@faculty.chiba-u.jp

## Abstract

A direct connection between two sequences of points, one of which is generated by seed mutations of the cluster algebra of type  $A_1^{(1)}$  and the other by time evolutions of the periodic discrete Toda lattice, is explicitly given. In this construction, each of them is realized as an orbit of a QRT map, and specialization of the parameters in the maps and appropriate choices of the initial points relate them. The connection with the periodic discrete Toda lattice enables us a geometric interpretation of the seed mutations of the cluster algebra of type  $A_1^{(1)}$  as an addition of points on an elliptic curve.

## 1 Introduction

A cluster algebra  $\mathcal{A}$ , which was firstly introduced by Fomin and Zelevinsky in 2002 [1], is a subalgebra of a field  $\mathcal{F}$  isomorphic to the field of rational functions in  $n$  independent variables ( $n$  is called the rank of  $\mathcal{A}$ ). Since their introduction cluster algebras have found applications in the field of dynamical systems such as  $Y$ -systems, discrete soliton equations and discrete Painlevé equations [3, 8, 9, 10, 12, 15, 13]. In this paper, we proceed to study links of cluster algebras and discrete integrable systems, and establish a direct connection between seed mutations of the cluster algebra of type  $A_1^{(1)}$  and time evolutions of the periodic discrete Toda lattice of the lowest dimension through their identifications with QRT maps. A QRT map is a member of

the paradigmatic family of two-dimensional integral maps found by Quispel, Roberts and Thompson in 1989 [16], which contains many two-dimensional reductions of discrete soliton equations and autonomous limits of the discrete Painlevé equations. Moreover, a QRT map is geometrically an addition of points on an elliptic curve called the invariant curve of the map [18]. Therefore, the connection with the periodic discrete Toda lattice gives a geometric interpretation of the seed mutations of the cluster algebra via the addition of points on the spectral curve of the Toda lattice. Since our method to associate a QRT map with a cluster algebra is applicable to any cluster algebra of rank 2, we will list corresponding map dynamical systems to a certain class of cluster algebras of rank 2.

Following [1, 2, 4], we briefly review some parts of cluster algebras necessary for our study. Let  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  be the set of generators of the ambient field  $\mathcal{F} = \mathbb{QP}(\mathbf{x})$ , where  $\mathbb{P} = (\mathbb{P}, \cdot, \oplus)$  is a semifield endowed with multiplication  $\cdot$  and auxiliary addition  $\oplus$  and  $\mathbb{QP}$  is the group ring of  $\mathbb{P}$  over  $\mathbb{Q}$ . Also let  $\mathbf{y} = (y_1, y_2, \dots, y_n)$  be an  $n$ -tuple in  $\mathbb{P}^n$  and  $B = (b_{ij})$  be an  $n \times n$  skew-symmetrizable integral matrix. The triple  $(\mathbf{x}, \mathbf{y}, B)$  is referred as the (labeled<sup>1</sup>) seed. We also refer to  $\mathbf{x}$  as the cluster of the seed, to  $\mathbf{y}$  as the coefficient tuple and to  $B$  as the exchange matrix. Each elements of  $\mathbf{x}$  and  $\mathbf{y}$  are called a cluster variable and a coefficient, respectively.

If an exchange matrix  $B$  is  $n \times n$  skew-symmetric then there uniquely corresponds a quiver  $Q = (Q_0, Q_1, s, t)$  through the relation

$$b_{ij} = \# \{ \text{arrows } i \rightarrow j \} - \# \{ \text{arrows } j \rightarrow i \},$$

where  $Q_0$  is the set of nodes (vertices) labeled by  $1, 2, \dots, n$ ,  $Q_1$  is the set of directed edges (arrows) and  $s$  (resp.  $t$ ) is the function which maps each arrow to its starting (resp. ending) vertex. We define a path of length  $l$  in  $Q$  to be a sequence of  $l$  arrows  $e_1, \dots, e_l$  with  $t(e_i) = s(e_{i+1})$  ( $1 \leq i < l$ ). We call a path of length  $l$  such that  $t(e_l) = s(e_1)$  an  $l$ -cycle. In particular, 1-cycle, i.e., an arrow  $e$  such that  $s(e) = t(e)$ , is called a loop. Thus, to each  $n \times n$  skew-symmetric matrix with integral entries there uniquely corresponds a quiver  $Q$  without loops or 2-cycles.

We introduce seed mutations. Let  $k \in [1, n]$  be an integer. The seed mutation  $\mu_k$  in the direction  $k$  transforms  $(\mathbf{x}, \mathbf{y}, B)$  into the seed  $\mu_k(\mathbf{x}, \mathbf{y}, B) =$ :

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<sup>1</sup>In this paper, we consider labeled seeds only. Therefore, we omit the term “labeled” hereafter.

$(\mathbf{x}', \mathbf{y}', B')$  defined as follows

$$b'_{ij} = \begin{cases} -b_{ij} & i = k \text{ or } j = k, \\ b_{ij} + [-b_{ik}]_+ b_{kj} + b_{ik} [b_{kj}]_+ & \text{otherwise,} \end{cases} \quad (1)$$

$$y'_j = \begin{cases} y_k^{-1} & j = k, \\ y_j y_k^{[b_{kj}]_+} (y_k \oplus 1)^{-b_{kj}} & j \neq k, \end{cases} \quad (2)$$

$$x'_j = \begin{cases} \frac{y_k \prod x_i^{[b_{ik}]_+} + \prod x_i^{[-b_{ik}]_+}}{(y_k \oplus 1) x_k} & j = k, \\ x_j & j \neq k, \end{cases} \quad (3)$$

where we define  $[a]_+ := \max(a, 0)$  for  $a \in \mathbb{Z}$ .

Let  $\mathbb{T}_n$  be the  $n$ -regular tree whose edges are labeled by  $1, 2, \dots, n$  so that the  $n$  edges emanating from each vertex receive different labels. We write  $t \xrightarrow{k} t'$  to indicate that vertices  $t, t' \in \mathbb{T}_n$  are joined by an edge labeled by  $k$ . We assign a seed  $\Sigma_t = (\mathbf{x}_t, \mathbf{y}_t, B_t)$  to every vertex  $t \in \mathbb{T}_n$  so that the seeds assigned to the endpoints of any edge  $t \xrightarrow{k} t'$  are obtained from each other by the seed mutation in direction  $k$ . We refer the assignment  $\mathbb{T}_n \ni t \mapsto \Sigma_t$  to a cluster pattern. We write the elements of  $\Sigma_t$  as follows

$$\mathbf{x}_t = (x_{1;t}, \dots, x_{n;t}), \quad \mathbf{y}_t = (y_{1;t}, \dots, y_{n;t}), \quad B_t = (b_{ij}^t).$$

Given a cluster pattern  $\mathbb{T}_n \ni t \mapsto \Sigma_t$ , we denote the union of clusters of all seeds in the pattern by

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} \mathbf{x}_t = \{x_{i;t} \mid t \in \mathbb{T}_n, 1 \leq i \leq n\}.$$

The cluster algebra  $\mathcal{A}$  associated with a given cluster pattern is the  $\mathbb{ZP}$ -subalgebra of the ambient field  $\mathcal{F}$  generated by all cluster variables:  $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$ .

Let  $\text{Trop}(u_1, u_2, \dots, u_m)$  be an abelian multiplicative group freely generated by  $u_1, u_2, \dots, u_m$ . We define the auxiliary addition  $\oplus$  in  $\text{Trop}(u_1, u_2, \dots, u_m)$  by

$$\prod_j u_j^{a_j} \oplus \prod_j u_j^{b_j} = \prod_j u_j^{\min(a_j, b_j)}.$$

We call  $(\text{Trop}(u_1, u_2, \dots, u_m), \cdot, \oplus)$  a tropical semifield. We say that a cluster pattern  $t \mapsto \Sigma_t$  on  $\mathbb{T}_n$  or the corresponding cluster algebra  $\mathcal{A}$  has principal

(tropical) coefficients at a vertex  $t_0$  if  $\mathbb{P} = \text{Trop}(y_1, y_2, \dots, y_n)$  and  $\mathbf{y}_{t_0} = (y_1, y_2, \dots, y_n)$ .

This paper is organized as follows. In section 2, we introduce the periodic discrete Toda lattice of dimension 4, and rewrite it as a QRT map  $\varphi_{\text{TL}} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . In this formulation, we use the additive group structure on an elliptic curve which is equivalent to the time evolution of the Toda lattice. In section 3, we show that the composition  $\mu_2 \circ \mu_1$  of the seed mutations  $\mu_1, \mu_2$  of the cluster algebra of type  $A_1^{(1)}$  induces a QRT map  $\varphi_{\text{CA}} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ . In section 4, we give the main theorem concerning a direct connection between two QRT maps  $\varphi_{\text{TL}}$  and  $\varphi_{\text{CA}}$ . By using the theorem, we relate special solutions to the periodic discrete Toda lattice with the cluster variables of type  $A_1^{(1)}$  obtained from a certain initial seed. Section 5 is devoted to concluding remarks. Finally, in appendix A, we extend the correspondence between the cluster algebra of type  $A_1^{(1)}$  and the QRT map, and list corresponding map dynamical systems to all cluster algebras of rank 2 associated with finite or affine Lie algebras.

## 2 A QRT map reduced from the periodic discrete Toda lattice

Let us consider a four-dimensional map  $\chi_1 : \mathbb{C}^4 \rightarrow \mathbb{C}^4; (\mathbf{I}, \mathbf{V}) \mapsto (\bar{\mathbf{I}}, \bar{\mathbf{V}})$ , where we put

$$\mathbf{I} = (I_1, I_2), \quad \mathbf{V} = (V_1, V_2)$$

and the evolution is defined by

$$\begin{cases} \bar{I}_i + \bar{V}_{i-1} = I_i + V_i, \\ \bar{I}_i \bar{V}_i = I_{i+1} V_i, \end{cases} \quad (4)$$

for  $i = 1, 2$ . Here the subscripts are reduced modulo 2. This map is known as the periodic discrete Toda lattice of the lowest dimension [6]. One can solve the equation (4) for  $\bar{I}_i$  and  $\bar{V}_i$  as follows

$$\begin{cases} \bar{I}_i = \frac{I_i + V_i}{I_{i-1} + V_{i-1}} I_{i-1}, \\ \bar{V}_i = \frac{I_{i+1} + V_{i+1}}{I_i + V_i} V_i. \end{cases} \quad (5)$$

Then we find that this is nothing but the discrete Toda lattice of type  $A_1^{(1)}$  [17] firstly introduced by Hirota in 1977 [5].

Let  $2 \times 2$  matrices  $L$  and  $M$  be

$$L = \begin{pmatrix} I_2 + V_1 & 1 - I_1 V_1 / y \\ I_2 V_2 - y & I_1 + V_2 \end{pmatrix}, \quad M = \begin{pmatrix} I_2 & 1 \\ -y & I_1 \end{pmatrix},$$

where  $y$  is the spectral parameter. Then the Lax form of  $\chi_1$  is given by

$$\bar{L}M = ML.$$

By using the Lax matrix  $L$ , the spectral curve  $\gamma_1$  of  $\chi_1$  is defined to be

$$\begin{aligned} \gamma_1 &:= (f(x, y) = 0) \cup \{P_\infty, P'_\infty\} \subset \mathbb{P}^2, \\ f(x, y) &:= y \det(L + xE), \end{aligned}$$

where  $E$  is the  $2 \times 2$  identity matrix and  $P_\infty, P'_\infty$  are the points at infinity. Upon introduction of the homogeneous coordinate of  $\mathbb{P}^2$ ,  $(x, y) \mapsto [X, Y, Z] = [x, y, 1]$ , we have  $P_\infty = [1, 0, 0]$  and  $P'_\infty = [0, 1, 0]$ . For generic choice of  $\mathbf{I}$  and  $\mathbf{V}$ ,  $\gamma_1$  is an elliptic curve. The defining polynomial  $f(x, y)$  of the spectral curve  $\gamma_1$  is expanded as follows

$$\begin{aligned} f(x, y) &= y^2 + y(x^2 + c_1 x + c_0) + c_{-1}, \\ c_{-1} &= I_1 I_2 V_1 V_2, \quad c_0 = I_1 I_2 + V_1 V_2, \quad c_1 = I_1 + I_2 + V_1 + V_2. \end{aligned}$$

The coefficients  $c_{-1}, c_0, c_1$  are the conserved quantities of the map  $\chi_1$ .

Put

$$\varphi_1 = I_2 V_2 - y, \quad \varphi_2 = x + I_1 + V_2.$$

Then  $\varphi(x, y) = {}^t(\varphi_1, \varphi_2)$  is the eigenvector of  $L$  associated with the eigenvalue  $-x$ , i.e., the following holds:

$$(L + xE)\varphi(x, y) = 0.$$

We see that the solution

$$(x, y) = (-I_1 - V_2, I_2 V_2)$$

to the system of equations  $\varphi_1 = 0$  and  $\varphi_2 = 0$  uniquely determines the point  $P = (x, y)$  on the spectral curve  $\gamma_1$ .

Let the initial value of  $\chi_1$  be  $(\mathbf{I}^0, \mathbf{V}^0) \in \mathbb{C}^4$ . We refer the  $t$ -th iteration of the map  $\chi_1$  from  $(\mathbf{I}^0, \mathbf{V}^0)$  to

$$(\mathbf{I}^t, \mathbf{V}^t) = \underbrace{\chi_1 \circ \chi_1 \circ \cdots \circ \chi_1}_t(\mathbf{I}^0, \mathbf{V}^0)$$

for  $t \geq 0$ . Let  $P^t$  be the point on  $\gamma_1$  given by

$$P^t = (x^t, y^t) = (-I_1^t - V_2^t, I_2^t V_2^t)$$

for  $t \geq 0$ . The sequence  $\{P^t\}_{t \geq 0}$  of points on  $\gamma_1$  is generated by successive applications of the map  $\chi_1$  to  $P^0$ .

On the other hand, it is known that the evolution  $P^t \mapsto P^{t+1}$  is interpreted as an addition of points on  $\gamma_1$  [11, 14].

**Theorem 1** *Let  $T$  be the point  $(-c_1, b)$  on  $\gamma_1$ , where  $b := -V_1^0 V_2^0$ . Then we have*

$$P^{t+1} = P^t + T$$

for  $t \geq 0$ . Here we choose  $P_\infty$  as the unit of addition. Since  $b$  is a conserved quantity of  $\chi_1$  as well as  $c_1$ ,  $T$  is a fixed point on the curve  $\gamma_1$ .  $\square$

Now we transform the map  $\chi_1$  into a QRT map in terms of the additive group structure of  $\gamma_1$  [18]. Let us shift  $T$  to the intersection point  $[0, 0, 1]$  of the axes  $X = 0$  and  $Y = 0$  of the plane  $\mathbb{P}^2$ . By employing the transformation  $\rho : (x, y) \mapsto (x + c_1, y - b)$  on  $\mathbb{C}^2$ , we have  $\rho(T) = (0, 0)$  and

$$(f \circ \rho^{-1})(x, y) = y^2 + y(x^2 - c_1 x + a) + b x(x - c_1),$$

where we put  $a := I_1^0 I_2^0 - V_1^0 V_2^0$ , which is also a conserved quantity of  $\chi_1$  because we have  $a^2 = c_0^2 - 4c_{-1}$ . These are written by using the homogeneous coordinate  $(x, y) \mapsto [X, Y, Z] = [x, y, 1]$  as follows

$$\begin{aligned} \rho(T) &\mapsto [0, 0, 1], \\ (f \circ \rho^{-1})(x, y) &\mapsto Y^2 Z + Y(X^2 - c_1 X Z + a Z^2) + b X Z(X - c_1 Z). \end{aligned}$$

Thus, the curve

$$\tilde{\gamma}_1 := ((f \circ \rho^{-1})(x, y) = 0) \cup \{P_\infty, P'_\infty\}$$

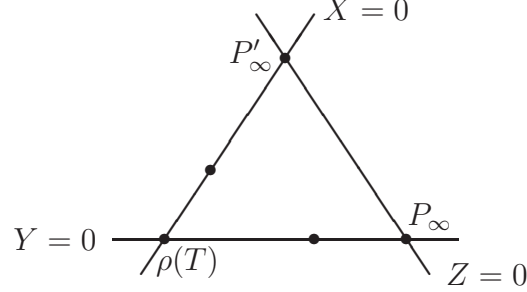


Figure 1: Configuration of the intersection points of  $\gamma_1$ ,  $X = 0$ ,  $Y = 0$  and  $Z = 0$  on the plane  $\mathbb{P}^2$ .

passes through the three intersection points  $\rho(T)$ ,  $P_\infty$ , and  $P'_\infty$  of the axes  $X = 0$ ,  $Y = 0$  and  $Z = 0$  of the plane  $\mathbb{P}^2$  (see figure 1). The intersection multiplicities of  $\tilde{\gamma}_1$  and  $\{X = 0\} \cup \{Y = 0\} \cup \{Z = 0\}$  at  $\rho(T)$ ,  $P_\infty$  and  $P'_\infty$  are two, two and three, respectively.

By employing the change of coordinate  $\sigma : \mathbb{P}^2 \rightarrow \mathbb{P}^2; [X, Y, Z] \mapsto [U, V, W] = [Z, X, Y]$ , we obtain

$$(\sigma \circ \rho)(T) = [1, 0, 0] =: \mathcal{T}, \quad \sigma(P_\infty) = [0, 1, 0] =: \mathcal{O}, \quad \sigma(P'_\infty) = [0, 0, 1].$$

In this coordinate,  $\tilde{\gamma}_1$  has the following form

$$\tilde{\gamma}_1 = (bxy^2 + y^2 - bc_1x^2y - c_1xy + ax^2 + x = 0) \cup \{\mathcal{O}, \mathcal{T}, \mathcal{S}\},$$

where  $\mathcal{S} := [c_1, 1, 0]$  and we use the inhomogeneous coordinate  $[U, V, W] \mapsto (x, y) = (U/W, V/W)$ . We find that the sequence  $\{P^t\}_{t \geq 0}$  of points on  $\gamma_1$  is mapped into the sequence  $\{\tilde{P}^t\}_{t \geq 0}$  of points on  $\tilde{\gamma}_1$ , where we put

$$\tilde{P}^t = \left( \frac{1}{V_2^t(I_2^t + V_1^t)}, \frac{1}{V_2^t} \right).$$

We then have the following theorem.

**Theorem 2** *Suppose that  $\mathcal{O}$  is the unit of addition on  $\tilde{\gamma}_1$ . Then we have*

$$\tilde{P}^{t+1} = \tilde{P}^t + \mathcal{T} \tag{6}$$

for  $t \geq 0$ . □

The time evolution  $\tilde{P}^t \mapsto \tilde{P}^{t+1}$  of points on  $\tilde{\gamma}$  reduced from  $\chi_1$  is geometrically realized in terms of intersections of  $\tilde{\gamma}_1$  with two lines  $l_1$  and  $l_2$  in the following manner. Let the line passing through  $\tilde{P}^t$  and parallel to the axis  $y = 0$  be  $l_1$ :

$$l_1 = (V_2^t y - 1 = 0).$$

Then  $l_1$  intersects  $\tilde{\gamma}_1$  at another point  $\tilde{Q}^t$ :

$$\tilde{Q}^t = \left( \frac{1}{V_2^t (I_1^t + V_1^t)}, \frac{1}{V_2^t} \right).$$

Also let the line passing through  $\tilde{Q}^t$  and parallel to the axis  $x = 0$  be  $l_2$ :

$$l_2 = (V_2^t (I_1^t + V_1^t) x - 1 = 0).$$

Then  $l_2$  intersects  $\tilde{\gamma}_1$  at  $\tilde{P}^{t+1}$ . This is a geometric realization of the time evolution  $\tilde{P}^t \mapsto \tilde{P}^{t+1}$ , and this gives a QRT map.

Actually, we observe that the map  $\tilde{P}^t \mapsto \tilde{P}^{t+1}$  is the QRT map  $\varphi_{\text{TL}} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given by the following matrices [16]

$$A_{\text{TL}} = \begin{pmatrix} 0 & 0 & a \\ b & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad B_{\text{TL}} = \begin{pmatrix} 0 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad (7)$$

let  $f_i$  and  $g_i$  ( $i = 1, 2, 3$ ) be

$$\begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} = A_{\text{TL}} \begin{pmatrix} (y^t)^2 \\ y^t \\ 1 \end{pmatrix} \times B_{\text{TL}} \begin{pmatrix} (y^t)^2 \\ y^t \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} g_1 \\ g_2 \\ g_3 \end{pmatrix} = {}^t A_{\text{TL}} \begin{pmatrix} (x^{t+1})^2 \\ x^{t+1} \\ 1 \end{pmatrix} \times {}^t B_{\text{TL}} \begin{pmatrix} (x^{t+1})^2 \\ x^{t+1} \\ 1 \end{pmatrix}.$$

We then obtain the QRT map  $\varphi_{\text{TL}} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1; (x^t, y^t) \mapsto (x^{t+1}, y^{t+1})$  as follows

$$x^{t+1} = \frac{f_1 - f_2 x^t}{f_2 - f_3 x^t} = \frac{-(bx^t + 1)(y^t)^2}{b(bx^t + 1)(y^t)^2 - (a - b)x^t},$$

$$y^{t+1} = \frac{g_1 - g_2 y^t}{g_2 - g_3 y^t} = \frac{a(x^{t+1})^2 + x^{t+1}}{(bx^{t+1} + 1)y^t}.$$



The invariant curve of  $\varphi_{\text{TL}}$  is also obtained:

$$(x^2 \ x \ 1) (A_{\text{TL}} + \lambda B_{\text{TL}}) \begin{pmatrix} y^2 \\ y \\ 1 \end{pmatrix} = bxy^2 + y^2 + b\lambda x^2 y + \lambda xy + ax^2 + x = 0, \quad (8)$$

where  $\lambda$  is the conserved quantity of  $\varphi_{\text{TL}}$ . This is the affine part of  $\tilde{\gamma}_1$  with imposing  $c_1 = -\lambda$ .

If we set  $\tilde{P}^t = (x^t, y^t)$  we have

$$x^t = \frac{1}{V_2^t (I_2^t + V_1^t)}, \quad y^t = \frac{1}{V_2^t}. \quad (9)$$

Then the equivalence of  $\chi_1$  and  $\varphi_{\text{TL}}$  is shown as follows. Given initial values  $\mathbf{I}^0, \mathbf{V}^0$  of  $\chi_1$ , the values of the parameters  $a$  and  $b$  of  $\varphi_{\text{TL}}$  are fixed:

$$a = I_1^0 I_2^0 - V_1^0 V_2^0, \quad b = -V_1^0 V_2^0.$$

The initial point  $(x^0, y^0)$  on the plane  $\mathbb{P}^1 \times \mathbb{P}^1$  is determined by (9) with imposing  $t = 0$ . Note that the initial values  $\mathbf{I}^0, \mathbf{V}^0$  of  $\chi_1$  must be chosen so that the initial point  $(x^0, y^0)$  is not a base point of the pencil  $\{\tilde{\gamma}\}_{\lambda \in \mathbb{P}^1}$ . Then  $\varphi_{\text{TL}}$  generates a sequence of points starting from  $(x^0, y^0)$  on the invariant curve (8). Conversely, if the values of the parameters  $a, b$  and the initial point  $(x^0, y^0)$  of  $\varphi_{\text{TL}}$  are given then we obtain the initial values  $\mathbf{I}^0, \mathbf{V}^0$  of  $\chi_1$  through

$$I_1^t = \frac{(bx^t + 1)y^t}{x^t}, \quad I_2^t = \frac{(a - b)x^t}{(bx^t + 1)y^t}, \quad V_1^t = -by^t, \quad V_2^t = \frac{1}{y^t} \quad (10)$$

with imposing  $t = 0$ . Then  $\chi_1$  generates a sequences of points starting from  $\mathbf{I}^0, \mathbf{V}^0$  on  $\mathbb{C}^4$ . Thus, we identify the maps  $\chi_1$  and  $\varphi_{\text{TL}}$ .

Note that the conserved quantity  $\lambda$  of  $\varphi_{\text{TL}}$  is recovered from  $\mathbf{I}^t$  and  $\mathbf{V}^t$  via their conserved quantity  $c_1$ :

$$\begin{aligned} \lambda = -c_1 &= -I_1^t - I_2^t - V_1^t - V_2^t \\ &= -\frac{(bx^t + 1)y^t}{x^t} - \frac{(a - b)x^t}{(bx^t + 1)y^t} + by^t - \frac{1}{y^t}. \end{aligned}$$

### 3 Mutations of the cluster algebra of type $A_1^{(1)}$ and a QRT map

Now we introduce the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$  which is of rank 2 and has principal coefficients. Let the initial seed  $\Sigma_0 = (\mathbf{x}_0, \mathbf{y}_0, B_0)$  be

$$\mathbf{x}_0 = (x_1, x_2), \quad \mathbf{y}_0 = (y_1, y_2), \quad B_0 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

where  $\mathbf{x}_0$  is the cluster,  $\mathbf{y}_0$  is the coefficient tuple and  $B_0$  is the exchange matrix. Also let the semifield of coefficients be  $\mathbb{P} = \text{Trop}(y_1, y_2)$ , the tropical semifield generated by  $y_1$  and  $y_2$ . Since the Cartan counterpart  $A(B_0)$  [2] of  $B_0$  is the Cartan matrix of type  $A_1^{(1)}$ :

$$A(B_0) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix},$$

we call the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$ . The Dynkin diagram of type  $A_1^{(1)}$  and the quiver associated with the exchange matrix  $B_0$  are given in figure 2.



Figure 2: The Dynkin diagram of type  $A_1^{(1)}$  (left) and the quiver associated with the exchange matrix  $B_0$  (right).

We consider the regular binary tree  $\mathbb{T}_2$  whose edges are labeled by the numbers 1 and 2. The tree  $\mathbb{T}_2$  is an infinite chain (see figure 3).

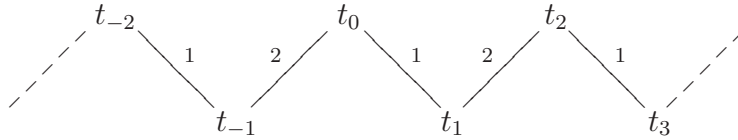


Figure 3: The regular binary tree  $\mathbb{T}_2$ .

Let  $T_2 \ni t_m \mapsto \Sigma_m = (\mathbf{x}_m, \mathbf{y}_m, B_m)$  be a cluster pattern of the cluster algebra  $\mathcal{A}$ . We see from (1)

$$B_m = \begin{cases} B_0 & m \text{ even,} \\ -B_0 & m \text{ odd.} \end{cases}$$

By applying (2) repeatedly, we find that the coefficients  $y_{1;m}$  and  $y_{2;m}$  for  $m \geq 2$  are the following monomials in the initial coefficients  $y_1$  and  $y_2$ , respectively:

$$y_{1;m} = \begin{cases} (y_1)^{-(m-1)} (y_2)^{-(m-2)} & m \text{ even,} \\ (y_1)^{m-2} (y_2)^{m-3} & m \text{ odd,} \end{cases}$$

$$y_{2;m} = \begin{cases} (y_1)^{m-2} (y_2)^{m-3} & m \text{ even,} \\ (y_1)^{-(m-1)} (y_2)^{-(m-2)} & m \text{ odd.} \end{cases}$$

We now consider the map

$$\mathbf{x}_m = (x_{1;m}, x_{2;m}) \xrightarrow{\mu_1} \mathbf{x}_{m+1} = (x_{1;m+1}, x_{2;m+1}) \xrightarrow{\mu_2} \mathbf{x}_{m+2} = (x_{1;m+2}, x_{2;m+2})$$

generated by successive application of the seed mutations  $\mu_1$  and  $\mu_2$ . We then find that the first map  $\mu_1$  fixes  $x_{2;m}$  and the second one  $\mu_2$   $x_{1;m+1}$ , and this is the same property as the QRT map. Therefore, we can realize the map  $\mu_2 \circ \mu_1$  on  $\mathbb{P}^1 \times \mathbb{P}^1$  as a QRT map.

**Proposition 1** *The composition  $\mu_2 \circ \mu_1$  of the seed mutations  $\mu_1$  and  $\mu_2$  of the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$  induces the QRT map given by the matrices*

$$A_{\text{CA}} = \begin{pmatrix} 0 & 0 & y_1^2 y_2^3 \\ 0 & 0 & 0 \\ y_1 y_2^2 & 0 & 1 \end{pmatrix}, \quad B_{\text{CA}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (11)$$

(Proof) Let us introduce new variables  $z^t$  and  $w^t$ :

$$z^t := \frac{x_{1;2t}}{(y_1 y_2)^t}, \quad w^t := \frac{x_{2;2t}}{(y_1 y_2)^t} \quad (t \geq 1), \quad z^0 := \frac{x_1}{y_1 (y_2)^2}, \quad w^0 := \frac{x_2}{y_2}.$$

Then the map  $\mu_2 \circ \mu_1 : (x_{1;m}, x_{2;m}) \mapsto (x_{1;m+2}, x_{2;m+2})$ , which is explicitly given by

$$x_{1;m+2} = \frac{(x_{2;m})^2 + y_1^{m-1} y_2^{m-2}}{x_{1;m}}, \quad x_{2;m+2} = \frac{(x_{1;m+2})^2 + y_1^m y_2^{m-1}}{x_{2;m}} \quad (m \geq 2),$$

$$x_{1;2} = \frac{y_1 (x_2)^2 + 1}{x_1}, \quad x_{2;2} = \frac{y_2 (x_{1;2})^2 + 1}{x_2},$$

reduces to  $(z^t, w^t) \mapsto (z^{t+1}, w^{t+1})$ ;

$$z^{t+1} = \frac{y_1 y_2^2 (w^t)^2 + 1}{y_1^2 y_2^3 z^t}, \quad w^{t+1} = \frac{y_1^2 y_2^3 (z^{t+1})^2 + 1}{y_1 y_2^2 w^t}.$$

It is easy to check that this is the QRT map  $\varphi_{\text{CA}} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$  given by the matrices (11).  $\square$

Hence,  $\varphi_{\text{CA}}$  produces a sequence  $\{(z^t, w^t)\}_{t \geq 0}$  of points on the invariant curve

$$y_1 y_2^2 w^2 + \lambda z w + y_1^2 y_2^3 z^2 + 1 = 0 \quad (12)$$

of the QRT map.

The above procedure reducing the QRT map  $\varphi_{\text{CA}}$  from the seed mutations of the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$  can be applied to any cluster algebra of rank 2. We will list all map dynamical systems reduced from the cluster algebras of rank 2 associated with finite or affine Lie algebras in appendix.

It is well known that the cluster variables of any cluster algebra are Laurent polynomials in the initial cluster variables, and this property is called the Laurent phenomenon [2]. By using the invariant curve (12) of  $\varphi_{\text{CA}}$ , we obtain an invariant of the seed mutations of the cluster algebra  $\mathcal{A}$  which exhibits the Laurent phenomenon.

**Proposition 2** *Let  $\mathbb{T}_2 \ni t_n \mapsto (x_{1;n}, x_{2;n})$  be a cluster pattern of the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$  for the initial seed  $\Sigma_0 = (\mathbf{x}_0, \mathbf{y}_0, B_0)$ :*

$$\mathbf{x}_0 = (x_1, x_2), \quad \mathbf{y}_0 = (y_1, y_2), \quad B_0 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

*Then the following*

$$\frac{y_1 y_2^2 (x_{2;n})^2 + y_1^2 y_2^3 (x_{1;n})^2 + (y_1 y_2)^{2n}}{x_{1;n} x_{2;n}}$$

*is the Laurent polynomial in the initial cluster variables  $x_1$  and  $x_2$  at any vertex  $t_n \in \mathbb{T}_2$ :*

$$\frac{y_1 y_2^2 (x_2)^2 + y_1^2 y_2^3 (x_1)^2 + 1}{x_1 x_2}.$$

(Proof) Since (12) is the invariant curve of the QRT map  $\varphi_{\text{CA}}$ , the conserved quantity  $\lambda$  of  $\varphi_{\text{CA}}$  is given by the Laurent polynomial in  $z$  and  $w$ :

$$\lambda = -\frac{y_1 y_2^2 w^2 + y_1^2 y_2^3 z^2 + 1}{zw}.$$

The statement is obvious by virtue of this fact.  $\square$

Geometrically, a QRT map is an addition of points on its invariant curve when it is an elliptic one. Unfortunately, the invariant curve (12) of the QRT map  $\varphi_{\text{CA}}$  is not an elliptic curve because it is quadratic. In the following section, we show that  $\varphi_{\text{CA}}$  is a degenerate limit of the QRT map  $\varphi_{\text{TL}}$  arising from the periodic discrete Toda lattice  $\chi_1$  and how the elliptic curve  $\tilde{\gamma}_1$ , the spectral curve of  $\chi_1$ , degenerates into the quadratic curve (12).

## 4 From the discrete Toda lattice to the cluster algebra via QRT maps

Let us observe the two sets of matrices (7) and (11), the former is induced from the discrete Toda lattice and the latter from the cluster algebra both of which are of type  $A_1^{(1)}$ . We then find that if  $b = 0$  the matrices  $B$ 's coincide with each other, while  $A$ 's are a bit different:

$$A_{\text{TL}} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_{\text{CA}} = \begin{pmatrix} 0 & 0 & y_1^2 y_2^3 \\ 0 & 0 & 0 \\ y_1 y_2^2 & 0 & 1 \end{pmatrix}.$$

For the QRT maps  $\varphi_{\text{TL}}$  and  $\varphi_{\text{CA}}$  respectively associated with  $\{A_{\text{TL}}, B_{\text{TL}}\}$  and  $\{A_{\text{CA}}, B_{\text{CA}}\}$ , an appropriate choice of  $y_1$  and  $y_2$  relates them.

**Theorem 3** *Let  $\xi, \eta, a \in \mathbb{C}$  and*

$$b = 0. \tag{13}$$

*Suppose*

$$y_1 = a^2 \xi \quad \text{and} \quad y_2 = 1/a\xi. \tag{14}$$

*Then we have*

$$\xi \varphi_{\text{TL}}^{2n}(\xi, \eta) = x^{(n)} \varphi_{\text{CA}}^n(\xi, \eta) \tag{15}$$

*for  $n \geq 0$ , where  $x^{(n)}$  is the  $x$ -component of  $\varphi_{\text{CA}}^n(\xi, \eta)$ .*

(Proof) Suppose  $b = 0$  then the QRT map  $\varphi_{\text{TL}} : (x^t, y^t) \mapsto (x^{t+1}, y^{t+1})$  induced from the periodic discrete Toda lattice has the following form

$$x^{t+1} = \frac{(y^t)^2}{ax^t}, \quad y^{t+1} = \frac{a(x^{t+1})^2 + x^{t+1}}{y^t}.$$

The iteration  $\varphi_{\text{TL}}^2 : (x^t, y^t) \mapsto (x^{t+2}, y^{t+2})$  of  $\varphi_{\text{TL}}$  leads to

$$x^{t+2} = \frac{1}{x^t} \left\{ \frac{(y^t)^2 + x^t}{ax^t} \right\}^2, \quad y^{t+2} = \frac{a(x^t)^2 x^{t+2} (ax^{t+2} + 1)}{y^t \{(y^t)^2 + x^t\}}. \quad (16)$$

Let  $x^{(n)}$  and  $y^{(n)}$  be the  $x$  and  $y$ -components of  $\varphi_{\text{CA}}^n(\xi, \eta)$ , respectively. Also let  $x^{\{n\}}$  and  $y^{\{n\}}$  be the  $x$  and  $y$ -components of  $\varphi_{\text{TL}}^n(\xi, \eta)$ , respectively.

We show (15) in terms of induction on  $n$ . Substituting  $t = 0$ ,  $x^{\{0\}} = \xi$  and  $y^{\{0\}} = \eta$  into (16), we obtain

$$\xi x^{\{2\}} = \left( \frac{\eta^2 + \xi}{a\xi} \right)^2$$

and

$$\xi y^{\{2\}} = \frac{a\xi^3 x^{\{2\}} (ax^{\{2\}} + 1)}{\eta(\eta^2 + \xi)} = \frac{\eta^2 + \xi}{a\xi} \times \frac{(\eta^2 + \xi)^2 + a\xi^3}{a\xi^2\eta}.$$

Similarly, for the QRT map  $\varphi_{\text{CA}} : (x^{(0)}, y^{(0)}) \mapsto (x^{(1)}, y^{(1)})$  induced from the seed mutations of the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$ , we have

$$x^{(1)} = \frac{(y^{(0)})^2 + y_1^{-1}y_2^{-2}}{y_1y_2x^{(0)}} = \frac{\eta^2 + \xi}{a\xi},$$

$$y^{(1)} = \frac{y_1y_2(x^{(1)})^2 + y_1^{-1}y_2^{-2}}{y^{(0)}} = \frac{a(x^{(1)})^2 + \xi}{\eta} = \frac{(\eta^2 + \xi)^2 + a\xi^3}{a\xi^2\eta}.$$

Thus, we obtain

$$\xi \varphi_{\text{TL}}^2(\xi, \eta) = x^{(1)} \varphi_{\text{CA}}(\xi, \eta).$$

Assume that (15) is true for  $n$ :

$$\xi x^{\{2n\}} = (x^{(n)})^2, \quad \xi y^{\{2n\}} = x^{(n)} y^{(n)}.$$

We then have

$$\begin{aligned}
\xi x^{\{2n+2\}} &= \frac{\xi}{x^{\{2n\}}} \left\{ \frac{(y^{\{2n\}})^2 + x^{\{2n\}}}{ax^{\{2n\}}} \right\}^2 \\
&= \frac{\xi}{\frac{(x^{(n)})^2}{\xi}} \frac{\left( \frac{x^{(n)}y^{(n)}}{\xi} \right)^4 + 2 \left( \frac{x^{(n)}y^{(n)}}{\xi} \right)^2 \frac{(x^{(n)})^2}{\xi} + \left( \frac{(x^{(n)})^2}{\xi} \right)^2}{a^2 \left( \frac{(x^{(n)})^2}{\xi} \right)^2} \\
&= \left\{ \frac{(y^{(n)})^2 + \xi}{a(x^{(n)})} \right\}^2 = (x^{(n+1)})^2.
\end{aligned}$$

Also we have

$$\begin{aligned}
\xi y^{\{2n+2\}} &= \frac{\xi a (x^{\{2n\}})^2 x^{\{2n+2\}} (ax^{\{2n+2\}} + 1)}{y^{\{2n\}} \left\{ (y^{\{2n\}})^2 + x^{\{2n\}} \right\}} \\
&= \frac{\xi a \frac{(x^{(n)})^4}{\xi^2} \frac{(x^{(n+1)})^2}{\xi} \left( a \frac{(x^{(n+1)})^2}{\xi} + 1 \right)}{\frac{x^{(n)}y^{(n)}}{\xi} \left\{ \left( \frac{x^{(n)}y^{(n)}}{\xi} \right)^2 + \frac{(x^{(n)})^2}{\xi} \right\}} \\
&= x^{(n+1)} \times \frac{ax^{(n)}x^{(n+1)}}{(y^{(n)})^2 + \xi} \times \frac{a(x^{(n+1)})^2 + \xi}{y^{(n)}} = x^{(n+1)}y^{(n+1)}.
\end{aligned}$$

This completes the proof.  $\square$

The cluster variables of  $\mathcal{A}$  are explicitly given by using the dependent variables  $\mathbf{I}$  and  $\mathbf{V}$  of the periodic discrete Toda lattice.

**Corollary 1** *Let  $\mathbb{T}_2 \ni t_m \mapsto \Sigma_m = (\mathbf{x}_m, \mathbf{y}_m, B_m)$  be the cluster pattern of the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$  for the initial seed  $\Sigma_0 = (\mathbf{x}_0, \mathbf{y}_0, B_0)$*

$$\mathbf{x}_0 = (x_1, x_2), \quad \mathbf{y}_0 = (y_1, y_2), \quad B_0 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

We moreover assume

$$x_1 = 1.$$

Let  $I_2^{2n}$  and  $V_2^{2n}$  be the  $2n$ -th solutions to the discrete Toda lattice  $\chi_1$  of type  $A_1^{(1)}$  starting from the initial values

$$I_1^0 = \frac{x_1}{x_2}, \quad I_2^0 = \frac{y_1 y_2 x_2}{x_1}, \quad V_1^0 = 0, \quad V_2^0 = \frac{y_2}{x_2}. \quad (17)$$

Then the cluster  $\mathbf{x}_m = (x_{1;m}, x_{2;m})$  is given as follows

$$x_{1;m} = \sqrt{\frac{(I_1^0 I_2^0)^{2\overline{m}-1}}{I_2^{2\overline{m}} V_2^{2\overline{m}}}}, \quad x_{2;m} = \sqrt{\frac{(I_1^0 I_2^0)^{2\overline{m}-1} I_2^{2\overline{m}}}{V_2^{2\overline{m}}}},$$

where  $\overline{m} := \lceil \frac{m}{2} \rceil = \min \{n \in \mathbb{Z} \mid n \geq \frac{m}{2}\}$  and  $\underline{m} := \lfloor \frac{m}{2} \rfloor = \max \{n \in \mathbb{Z} \mid n \leq \frac{m}{2}\}$ . Here the square roots are so chosen as to satisfy

$$x_{1;2n} x_{2;2n} = \frac{(I_1^0 I_2^0)^{2n-1}}{V_2^{2n}}.$$

(Proof) Considering the initial values (17), we have

$$b = -V_1^0 V_2^0 = 0.$$

This satisfies the condition (13) on  $b$  in theorem 3. Since  $V_1^0 = 0$ , we see from (5) that  $V_1^n = 0$  holds for  $n \geq 0$ .

Noting (10), the initial value  $(x^{\{0\}}, y^{\{0\}})$  of the QRT map  $\varphi_{\text{TL}}$  induced from the Toda lattice is given as follows

$$x^{\{0\}} = \frac{1}{V_2^0 (I_2^0 + V_1^0)} = \frac{1}{I_2^0 V_2^0} = x_1^{(0)}, \quad y^{\{0\}} = \frac{1}{V_2^0} = x_2^{(0)},$$

where  $(x_1^{(0)}, x_2^{(0)}) = (x_1/y_1(y_2)^2, x_2/y_2)$  is the initial value of the QRT map  $\varphi_{\text{CA}}$ . Thus, by theorem 3, we have

$$\begin{aligned} (x^{(n)})^2 &= \left( \frac{x_{1;2n}}{(y_1 y_2)^n} \right)^2 = x_1^{(0)} x^{\{2n\}} = \frac{x_1^{(0)}}{I_2^{2n} V_2^{2n}}, \\ x^{(n)} y^{(n)} &= \frac{x_{1;2n}}{(y_1 y_2)^n} \frac{x_{2;2n}}{(y_1 y_2)^n} = x_1^{(0)} y^{\{2n\}} = \frac{x_1^{(0)}}{V_2^{2n}}, \end{aligned}$$

and hence have

$$\left( \frac{x_{2;2n}}{(y_1 y_2)^n} \right)^2 = \frac{x_1^{(0)} I_2^{2n}}{V_2^{2n}}.$$



Here we use  $V_1^{2n} = 0$  and (10) again. The relations

$$x_1^{(0)} = \frac{1}{I_2^0 V_2^0}, \quad y_1 y_2 = I_1^0 I_2^0, \quad x_{1;2n} = x_{1;2n-1}, \quad x_{2;2n} = x_{2;2n+1}$$

leads to the conclusion.  $\square$

The following corollary is similarly shown.

**Corollary 2** *Let the initial values of the discrete Toda lattice  $\chi_1$  of type  $A_1^{(1)}$  be*

$$I_1^0, \quad I_2^0, \quad V_1^0 = 0, \quad V_2^0.$$

Also let  $\mathbb{T}_2 \ni t_m \mapsto \Sigma_m = (\mathbf{x}_m, \mathbf{y}_m, B_m)$  be the cluster pattern of the cluster algebra  $\mathcal{A}$  of type  $A_1^{(1)}$  for the initial seed  $\Sigma_0 = (\mathbf{x}_0, \mathbf{y}_0, B_0)$

$$\mathbf{x}_0 = (x_1, x_2) = \left(1, \frac{1}{I_1^0}\right), \quad \mathbf{y}_0 = (y_1, y_2) = \left(y_1, \frac{I_1^0 I_2^0}{y_1}\right),$$

$$B_0 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}.$$

Then the solution  $(\mathbf{I}^{2n}, \mathbf{V}^{2n})$  to  $\chi_1$  is given as follows

$$I_1^{2n} = y_1 y_2 \frac{x_{1;2n}}{x_{2;2n}}, \quad I_2^{2n} = \frac{x_{2;2n}}{x_{1;2n}}, \quad V_1^{2n} = 0, \quad V_2^{2n} = \frac{(y_1)^{2n} (y_2)^{2n-1} (x_2)^2}{x_1 x_{1;2n} x_{2;2n}}.$$

$\square$

If we impose the condition (13) on the invariant curve  $\tilde{\gamma}$  (8) of the QRT map  $\varphi_{\text{TL}}$  then it reduces to

$$y^2 + \lambda xy + ax^2 + x = 0. \tag{18}$$

Let the  $n$ -th solution to  $\varphi_{\text{CA}}$  be  $(x^{(n)}, y^{(n)})$ . The  $2n$ -th solution  $(x^{\{2n\}}, y^{\{2n\}})$  to  $\varphi_{\text{TL}}$ , which is equivalent to  $((x^{(n)})^2/x^{\{0\}}, x^{(n)}y^{(n)}/x^{\{0\}})$  by theorem 3, solves (18). By substituting  $(x, y) = ((x^{(n)})^2/x^{\{0\}}, x^{(n)}y^{(n)}/x^{\{0\}})$  into (18), we have

$$(y^{(n)})^2 + \lambda x^{(n)} y^{(n)} + a (x^{(n)})^2 + x^{\{0\}} = 0$$

for  $x^{\{0\}} \neq 0$ . Since the invariant curve (12) of the QRT map  $\varphi_{\text{CA}}$  imposing the condition (14) reduces to

$$w^2 + \lambda \xi zw + az^2 + \xi = 0,$$

$(z, w) = (x^{(n)}, y^{(n)})$  solves it. Thus, we see that the invariant curve (12) of  $\varphi_{\text{CA}}$  is a singular member of the pencil  $\{\tilde{\gamma}\}_{c_1 \in \mathbb{P}^1}$  of the invariant curves  $\tilde{\gamma}$  of  $\varphi_{\text{TL}}$ . Moreover, addition (6) of points on  $\tilde{\gamma}$  induces the map  $\varphi_{\text{CA}}$  in the limit  $b \rightarrow 0$ .

## 5 Concluding remarks

We gave a new expression of the periodic discrete Toda lattice of dimension 4 as a QRT map by using its additive group structure equivalent to the time evolution of the Toda lattice. In this formulation, we transform the spectral curve of the Toda lattice into a bi-quadratic curve so that its additive group structure coincides with that of the invariant curve of the QRT map. We also realized the seed mutations of the cluster algebra of type  $A_1^{(1)}$  as a birational map on  $\mathbb{P}^1 \times \mathbb{P}^1$  and represented it as a QRT map. We then related the two QRT maps directly by specializing the parameters contained in them. This formulation gives a geometric interpretation of the seed mutations of the cluster algebra of type  $A_1^{(1)}$  as an addition of points on the elliptic curve. Using the direct connection, cluster variables of the cluster algebra of type  $A_1^{(1)}$  was given in terms of the solutions to the periodic discrete Toda lattice starting from appropriate initial variables, and vice versa.

Although we showed certain links of cluster algebras of rank 2 and QRT maps only in this paper, such phenomena can be found in cluster algebras of any rank because a seed mutation is essentially local, i.e., it depends on an exchange relation among neighboring cluster variables linked by the exchange matrix. Therefore, if we focus on two cluster variables, say  $x_k$  and  $x_l$ , appearing in an exchange relation and consider a composition of the seed mutations in the directions  $k$  and  $l$  then it can be regarded as a map dynamical system on the plane, which is the intersection of the hyperplanes given by the unmutated variables. By construction, a map dynamical system thus obtained depends only on the topology of the quiver associated with the exchange matrix. Moreover, successive application of seed mutations induces a global map dynamical system which is composition of the local ones. We

will report map dynamical systems arising from seed mutations of cluster algebras of general rank in a forthcoming paper.

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## A Cluster algebras of rank 2 and QRT maps

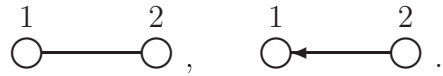
Let us consider the cluster algebra of type  $A_2$  whose initial seed  $\Sigma_0 = (\mathbf{x}_0, \mathbf{y}_0, B_0)$  is given as follows

$$\mathbf{x}_0 = (x_1, x_2), \quad \mathbf{y}_0 = (y_1, y_2), \quad B_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that we do not assume the semifield  $\mathbb{P}$  to be tropical. The Cartan counterpart  $A(B_0)$  of  $B_0$  is of type  $A_2$ :

$$A(B_0) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The Dynkin diagram and the quiver associated with  $B_0$  is given as follows



Let  $\mathbb{T}_2 \ni t_n \mapsto \Sigma_n = (\mathbf{x}_n, \mathbf{y}_n, B_n)$  be the cluster pattern of the cluster algebra. Consider the map  $\mathbf{x}_n \xrightarrow{\mu_1} \mathbf{x}_{n+1} \xrightarrow{\mu_2} \mathbf{x}_{n+2}$ . We then have

$$x_{1;n+2} = x_{1;n+1} = \frac{y_{1;n}x_{2;n} + 1}{(y_{1;n} \oplus 1)x_{1;n}} = y_{2;n+2} \frac{y_{1;n}x_{2;n} + 1}{\frac{x_{1;n}}{y_{2;n}}},$$

$$x_{2;n+2} = \frac{y_{2;n+1}x_{1;n+1} + 1}{(y_{2;n+1} \oplus 1)x_{2;n+1}} = \frac{y_{2;n+1}x_{1;n+2} + 1}{(y_{2;n+1} \oplus 1)x_{2;n}} = \frac{1}{y_{1;n+2}} \frac{\frac{x_{1;n+2}}{y_{2;n+2}} + 1}{y_{1;n}x_{2;n}},$$

where we use  $x_{1;n+1} = x_{1;n+2}$ ,  $x_{2;n} = x_{2;n+1}$  and the following relations obtained from the mutation of  $\mathbf{y}$ :

$$y_{1;n+2} = y_{1;n+1}(y_{2;n+1} \oplus 1), \quad y_{1;n}y_{1;n+1} = 1,$$

$$y_{2;n+1} = y_{2;n}(y_{1;n} \oplus 1), \quad y_{2;n+1}y_{2;n+2} = 1.$$

If we put

$$z^t = \frac{x_{1;2t}}{y_{2;2t}}, \quad w^t = y_{1;2t}x_{2;2t}$$

then we have

$$z^{t+1} = \frac{w^t + 1}{z^t}, \quad w^{t+1} = \frac{z^{t+1} + 1}{w^t}. \quad (19)$$

We see that the map  $(z^t, w^t) \mapsto (z^{t+1}, w^{t+1})$  is the QRT map given by the matrices

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The invariant curve of the QRT map is the following cubic curve

$$(z + 1)w^2 + (z^2 + \lambda z + 2)w + (z + 1)^2 = 0,$$

and  $\lambda$  is the conserved quantity of the map.

The QRT map (19) has period 5 for a generic initial point, which reflects the fact that the corresponding cluster algebra is of finite type having 5 independent cluster variables.

In the same manner, we obtain birational-map dynamical systems on  $\mathbb{P}^1 \times \mathbb{P}^1$  from the cluster algebras of type  $A_1 \times A_1$ ,  $B_2$ ,  $G_2$ ,  $A_1^{(1)}$  and  $A_2^{(2)}$ , which complete cluster algebras of rank 2 associated with finite or affine Lie algebras. Three of them ( $A_1 \times A_1$ ,  $B_2$  and  $A_1^{(1)}$ ) lead to QRT maps, while  $G_2$  and  $A_2^{(2)}$  induce dynamical systems of higher degree. We list them in the following. We use the same notations as in the case of type  $A_2$  unless otherwise stated.

- Type  $A_1 \times A_1$

Initial exchange matrix and its Cartan counterpart:

$$B_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A(B_0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Dynkin diagram and quiver associated with  $B_0$ :

$$\begin{array}{cc} 1 & 2 \\ \bigcirc & \bigcirc \end{array}, \quad \begin{array}{cc} 1 & 2 \\ \bigcirc & \bigcirc \end{array}.$$

Variable transformation:

$$z^t = x_{1;2t}, \quad w^t = x_{2;2t}.$$

QRT map and its matrices:

$$z^{t+1} = \frac{1 + y_1}{(1 \oplus y_1)z^t}, \quad w^{t+1} = \frac{1 + y_2}{(1 \oplus y_2)w^t},$$

$$A = \begin{pmatrix} 0 & 1 \oplus y_1 & 0 \\ 1 \oplus y_2 & 0 & 1 + y_2 \\ 0 & 1 + y_1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Invariant curve:

$$(1 \oplus y_2)zw^2 + \{(1 \oplus y_1)z^2 + \lambda z + 1 + y_1\}w + (1 + y_2)z = 0.$$

- Type  $B_2$

Initial exchange matrix and its Cartan counterpart:

$$B_0 = \begin{pmatrix} 0 & -2 \\ 1 & 0 \end{pmatrix}, \quad A(B_0) = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}.$$

Dynkin diagram (since  $B_0$  is not skew-symmetric, there is no quiver associated with  $B_0$ ):



Variable transformation:

$$z^t = \frac{x_{1;2t}}{\sqrt{y_{2;2t}}}, \quad w^t = y_{1;2t}x_{2;2t}.$$

QRT map and its matrices:

$$z^{t+1} = \frac{w^t + 1}{z^t}, \quad w^{t+1} = \frac{(z^{t+1})^2 + 1}{w^t},$$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Invariant curve:

$$w^2 + (z^2 + \lambda z + 2)w + z^2 + 1 = 0.$$

- Type  $G_2$

Initial exchange matrix and its Cartan counterpart:

$$B_0 = \begin{pmatrix} 0 & 1 \\ -3 & 0 \end{pmatrix}, \quad A(B_0) = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

Dynkin diagram:



Variable transformation:

$$z^t = y_{2;2t}x_{1;2t}, \quad w^t = \frac{x_{2;2t}}{\sqrt[3]{y_{1;2t}}}.$$

Map dynamical system (not a QRT map):

$$z^{t+1} = \frac{(w^t)^3 + 1}{z^t}, \quad w^{t+1} = \frac{z^{t+1} + 1}{w^t}.$$

Invariant curve:

$$\begin{aligned} z + w + \frac{1+z}{w} + \frac{(1+z)^3 + w^3}{zw^3} + \frac{(1+z)^2 + w^3}{zw^2} \\ + \frac{(1+z)^3 + 2w^3 + w^6 + 3zw^3}{z^2w^3} + \frac{1+z+w^3}{zw} + \frac{1+w^3}{z} = \lambda. \end{aligned}$$

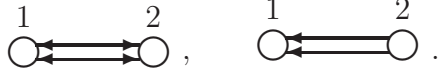
**Remark 1** *Since the cluster algebra of type  $G_2$  is of finite type having period 8, the sum of all 8 cluster variables is invariant under the seed mutation. This gives the above sextic invariant curve [7].*

- Type  $A_1^{(1)}$

Initial exchange matrix and its Cartan counterpart:

$$B_0 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}, \quad A(B_0) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}.$$

Dynkin diagram and quiver associated with  $B_0$ :



Variable transformation:

$$z^t = \frac{x_{1;2t}}{\sqrt{y_{2;2t}}}, \quad w^t = \sqrt{y_{1;2t}} x_{2;2t}.$$

QRT map and its matrices:

$$z^{t+1} = \frac{(w^t)^2 + 1}{z^t}, \quad w^{t+1} = \frac{(z^{t+1})^2 + 1}{w^t},$$

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (20)$$

Invariant curve:

$$w^2 + \lambda zw + z^2 + 1 = 0.$$

**Remark 2** *The QRT map (20) is independent of the choice of  $\mathbb{P}$ . Therefore, if we set  $y_1 = y_2 = 1$  in the tropical case (11) we obtain (20).*

- Type  $A_2^{(2)}$

Initial exchange matrix and its Cartan counterpart:

$$B_0 = \begin{pmatrix} 0 & -4 \\ 1 & 0 \end{pmatrix}, \quad A(B_0) = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}.$$

Dynkin diagram:



Variable transformation:

$$z^t = \frac{x_{1;2t}}{\sqrt[4]{y_{2;2t}}}, \quad w^t = y_{1;2t} x_{2;2t}.$$

Map dynamical system (not a QRT map):

$$z^{t+1} = \frac{w^t + 1}{z^t}, \quad w^{t+1} = \frac{(z^{t+1})^4 + 1}{w^t}.$$

The invariant curve has not been obtained yet (since the cluster algebra of type  $A_2^{(2)}$  is of infinite type, we can not apply the same procedure as in the case of type  $G_2$ ).

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